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On the motion of mechanical networks

GUY PAYRE
Mechanical Engineering Department, Université de Sherbrooke, Sherbrooke, Qué., J1K 2R1, Canada
(guy.payre@usherbrooke.ca)

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Abstract. This paper develops the differential equations governing the motion of spatial networks to which mechanical features such as masses, stiffness coefficients, tensions and bending moments have been associated. These networks generalize the concept of particle systems introduced for the simulation of flexible bodies and extend their application to elastic models. The network deformation is shown to be related to the internal tensions and moments by a set of vectors, the directors of the network. A numerical example describing a rotating flexible beam is presented.

Key words: discrete elasticity, large deformations, multibody dynamics, particle systems

1. Introduction

The main theories of rational mechanics are based on the concept of a continuous distribution of matter, which is called the continuum. This concept has been used extensively as an idealization to describe such objects as rigid bodies, deformable bodies and fluids. However, it has long been known that quite different (*i.e.*, discrete) approaches could be applied to obtain useful insights into the behavior of mechanical systems.

In the past years, access to increasingly high-performance computers has resulted in a strong incentive to revisit the fundamentals of this line of reasoning. The general-purpose numerical schemes such as finite-difference or finite-element methods are formulated directly from the partial differential equations (or their variational counterparts) governing the behavior of a continuum. On the other hand, the complexity of fluid motion, induced by a wide distribution of spatial scales needed to describe adequately the vorticity field, has led to a radically different point of view. Instead of trying to discretize a system of partial differential equations, we closely study a large number of individual particles of which each one is interacting potentially with all the others according to a given set of collision laws. First seen in the sixties, these methods were known as the Marker-And-Cell (MAC) or Particle-In-Cell (PIC) methods. They matured during the eighties into what is now known as the lattice-gas computational methods for fluid flows [1].

In the field of solid mechanics, this approach has been used mainly for computer animations in order to simulate the motion of flexible systems such as draperies or garments. A large number of particles are connected in a mesh, each particle interacting with its neighbors by the mean of internal forces [2]. The mechanical principles used in the setup of such models are usually crude or sometimes replaced by heuristics to speed up the computations [3]. In three-dimensional meshes, the internal forces are usually assumed to be exerted along the line joining the two interacting particles.

The purpose of this paper is to establish the differential equations describing the elastic motion of particle systems whereby the strain energy depends not only on the relative distances between individual particles but also on the relative orientation between the edges connecting the mass points. In order to do so, the network is endowed with a small set of "physical" constants, and a small number of variables are defined in order to mimic the main features of an elastic state. These quantities are then bound to interact according to some local behavior laws as if they were true mechanical quantities (lengths, forces, moments ...). From this process follow the global laws governing the evolution of the state variables associated with this mechanical network.

The structure of this paper is as follows: Section 2 introduces the notations and establishes some useful identities. The components of the network mechanical state vector (positions, velocities, tensions and bending moments) are defined in Section 3, and their laws of evolution are derived in Section 4, leading to the main differential system (42–45). A numerical example will be presented in Section 5 in order to illustrate how these concepts do extend the domain of applicability of particle systems to model elastic bodies.

2. Notations

A geometrical definition of the networks in motion will be presented first (see Figure 1). Let us denote by $\mathcal{N} = \{P_i(t)\}$ a set of nn moving points of $\mathbb{R}^3$ (t denotes the time). These points are called the *nodes* of the network. Their positions and velocities are respectively given by $\mathbf{X}_i(t)$ and $\mathbf{V}_i(t)$. A node is connected to its neighbors by a set of *edges*, defined as a couple (P_i, P_j) of nodes. The set of edges $\mathcal{E}$ is a subset of the Cartesian product $\mathcal{N} \times \mathcal{N}$. The whole topological network $\mathcal{G}$ is given by the couple $(\mathcal{N}, \mathcal{E})$, with $nn = \mathrm{Card}(\mathcal{N})$ and $ne = \mathrm{Card}(\mathcal{E})$.

To the edge p is assigned a couple of integers $[B(p), E(p)]$, with $B(p) < E(p)$. The first integer $B(p)$ is the index of the first vertex (node) of the edge (*Begin of* p), and $E(p)$ is the index of the second vertex (*End of* p). Let us denote by $\mathbf{S}_p$ the unit vector on the edge p, oriented from $B(p)$ toward $E(p)$. Following Ericksen and Truesdell [4], these unit vectors will be called the *primary directors* for the network $\mathcal{G}$.

To the node P_i are associated the sets $\mathrm{Ie}^+(i)$, $\mathrm{Ie}^-(i)$, $\mathrm{Ie}(i)$ and $\mathrm{Ie}^S(i)$ defined by the relations:

$$p \in \mathrm{Ie}^-(i) \Leftrightarrow i = B(p), \tag{1}$$

$$p \in \mathrm{Ie}^+(i) \Leftrightarrow i = E(p), \tag{2}$$

$$\mathrm{Ie}(i) = \mathrm{Ie}^-(i) \cup \mathrm{Ie}^+(i), \tag{3}$$

$$\mathrm{Ie}^S(i) = \{ (p, q) \quad p \in \mathrm{Ie}(i) \quad q \in \mathrm{Ie}(i) \quad p > q \}. \tag{4}$$

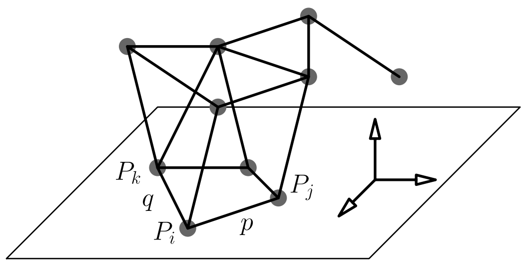

Figure 1. A mechanical network of mass points.

Let there be a node i and an edge $p \in \mathrm{Ie}(i)$. We introduce the following notations:

$$\ell_p = \|\mathbf{X}_{E(p)} - \mathbf{X}_{B(p)}\|, \tag{5}$$

$$a_i^p = \begin{cases} E(p) & \text{if } p \in \mathrm{Ie}^-(i), \\ B(p) & \text{if } p \in \mathrm{Ie}^+(i) \end{cases}$$

$$s_i^p = \begin{cases} +1 & \text{if } p \in \mathrm{Ie}^+(i), \\ -1 & \text{if } p \in \mathrm{Ie}^-(i) \end{cases}$$

$$\mathbf{V}^p = \mathbf{V}_{E(p)} - \mathbf{V}_{B(p)}. \tag{6}$$

Let us remark that, letting $s_i^p = 0$ for $p \notin \mathrm{Ie}(i)$, the length ℓ_p of an edge p, its primary director $\mathbf{S}_p$ and its relative velocity $\mathbf{V}^p$ could as well be defined by the relations (with $i = B(p)$ and $j = E(p)$)

$$\ell_p = \left\| \sum_k s_k^p \mathbf{X}_k \right\| = \left\| s_i^p \mathbf{X}_i + s_j^p \mathbf{X}_j \right\|,$$

$$\mathbf{S}_p = \left(\sum_k s_k^p \mathbf{X}_k \right) / \ell_p = \frac{s_i^p \mathbf{X}_i + s_j^p \mathbf{X}_j}{\ell_p},$$

$$\mathbf{V}^p = \sum_k s_k^p \mathbf{V}_k = s_i^p \mathbf{V}_i + s_j^p \mathbf{V}_j.$$

A couple of edges (p, q) sharing a vertex i is called an *elbow* of the node P_i, and will be denoted by the symbol $[i; (p, q)]$, with $p > q$.

Let us consider an elbow $[i; (p, q)]$. In the case where $\mathbf{S}_p \wedge \mathbf{S}_q \neq 0$ (*i.e.*, when $\mathbf{S}_p$ and $\mathbf{S}_q$ are not collinear), the secondary director $\mathbf{B}_{pq}$ is defined as

$$\mathbf{B}_{pq} = s_i^p s_i^q \frac{\mathbf{S}_p \wedge \mathbf{S}_q}{\|\mathbf{S}_p \wedge \mathbf{S}_q\|}. \tag{7}$$

In the case $\mathbf{S}_p \wedge \mathbf{S}_q = 0$, the director $\mathbf{B}_{pq}$ will be a unit vector normal to these two primary directors and will be defined by some other condition. The directors $(\mathbf{S}_p, \mathbf{S}_q)$ represent for the network the discrete equivalent of the unitary tangent vector $\mathbf{t}$ to the "curve" $(i, s_i^q \mathbf{S}_q, -s_i^p \mathbf{S}_p)$, as $\mathbf{B}_{pq}$ represents the discrete binormal $\mathbf{b}$ for this same curve.

Given two unit vectors $\mathbf{S}_p$ and $\mathbf{S}_q$, we denote by $\mathbf{S}^{pq}$ the matrix computed by

$$\mathbf{S}^{pq} = \mathbf{S}_p \cdot \mathbf{S}_q \mathbf{I} - \mathbf{S}_p \mathbf{S}_q^t. \tag{8}$$

Using these notations, we may obtain the temporal derivative of a primary director from the kinetics of the edge extremities by the relation:

$$\dot{\mathbf{S}}_p = \frac{1}{\ell_p} \mathbf{S}^{pp} \mathbf{V}^p. \tag{9}$$

Proof. The vector $\mathbf{S}_p$ is equal to

$$\mathbf{S}_p = \frac{1}{\ell_p} \left(\mathbf{X}_{E(p)} - \mathbf{X}_{B(p)} \right).$$

Let $i = B(p)$ and $j = E(p)$ $(i < j)$. Then,

$$\dot{\mathbf{S}}_p = \frac{1}{\ell_p} \left(\mathbf{V}_j - \mathbf{V}_i \right) - \frac{\dot{\ell}_p}{\ell_p^2} \left(\mathbf{X}_j - \mathbf{X}_i \right) = \frac{1}{\ell_p} \mathbf{V}^p - \frac{\dot{\ell}_p}{\ell_p} \mathbf{S}_p.$$

Let us then compute $\dot{\ell}_p$ by deriving with regard to time the two sides of the identity

$$\begin{aligned}
&\ell_p^2 = (\mathbf{X}_j - \mathbf{X}_i)\cdot(\mathbf{X}_j - \mathbf{X}_i)\,,\\
&2\,\ell_p \dot{\ell}_p = 2\,(\mathbf{X}_j - \mathbf{X}_i)\cdot(\mathbf{V}_j - \mathbf{V}_i)\,,\\
&\dot{\ell}_p = \mathbf{S}_p\cdot(\mathbf{V}_j - \mathbf{V}_i) = \mathbf{S}_p\cdot\mathbf{V}^p.
\end{aligned} \tag{10}$$

As a consequence,

$$\begin{aligned}
\dot{\mathbf{S}}_p &= \frac{1}{\ell_p}\left(\mathbf{V}^p - \mathbf{S}_p\cdot\mathbf{V}^p\,\mathbf{S}_p\right) = \frac{1}{\ell_p}\left(\mathbf{V}^p - \mathbf{S}_p\,\mathbf{S}_p^t\,\mathbf{V}^p\right)\\
&= \frac{1}{\ell_p}\left(\mathbf{I} - \mathbf{S}_p\mathbf{S}_p^t\right)\mathbf{V}^p = \frac{1}{\ell_p}\,\mathbf{S}^{pp}\,\mathbf{V}^p.
\end{aligned}$$

□

We shall need later the following three relations, in which $\mathbf{S}_p$ and $\mathbf{S}_q$ are two unit vectors:

$$\mathbf{S}^{pp}\mathbf{S}_q = \mathbf{S}_q - \mathbf{S}_p\cdot\mathbf{S}_q\,\mathbf{S}_p. \tag{11}$$

Proof. As $\mathbf{S}^{pp} = \mathbf{I} - \mathbf{S}_p\mathbf{S}_p^t$, we can write

$$\begin{aligned}
\mathbf{S}^{pp}\mathbf{S}_q &= \mathbf{S}_q - \mathbf{S}_p\mathbf{S}_p^t\mathbf{S}_q = \mathbf{S}_q - \mathbf{S}_p\left(\mathbf{S}_p^t\mathbf{S}_q\right)\\
&= \mathbf{S}_q - \mathbf{S}_p\cdot\mathbf{S}_q\,\mathbf{S}_p.
\end{aligned}$$

□

For any vector $\mathbf{M}$,

$$\mathbf{S}^{pp}\mathbf{M}\cdot\mathbf{S}_q = \mathbf{S}^{pp}\mathbf{S}_q\cdot\mathbf{M}. \tag{12}$$

Proof.

$$\begin{aligned}
\mathbf{S}^{pp}\mathbf{M}\cdot\mathbf{S}_q &= \left(\mathbf{M} - \mathbf{S}_p\mathbf{S}_p^t\mathbf{M}\right)\cdot\mathbf{S}_q = \mathbf{S}_q\cdot\mathbf{M} - \mathbf{S}_q^t\left(\mathbf{S}_p\mathbf{S}_p^t\mathbf{M}\right)\\
&= \mathbf{S}_q\cdot\mathbf{M} - (\mathbf{S}_p\cdot\mathbf{S}_q)\left(\mathbf{S}_p^t\mathbf{M}\right) = \mathbf{S}_q\cdot\mathbf{M} - (\mathbf{S}_p\cdot\mathbf{S}_q)\,(\mathbf{S}_p\cdot\mathbf{M})\\
&= \left[\mathbf{S}_q - (\mathbf{S}_p\cdot\mathbf{S}_q)\,\mathbf{S}_p\right]\cdot\mathbf{M}\\
&= \mathbf{S}^{pp}\mathbf{S}_q\cdot\mathbf{M} \qquad \text{(using (11))}.
\end{aligned}$$

□

Finally,

$$\mathbf{S}^{pp}\left(\mathbf{S}_p\wedge\mathbf{M}\right) = \mathbf{S}_p\wedge\mathbf{M}. \tag{13}$$

Proof. By definition, $\mathbf{S}^{pp} = \mathbf{I} - \mathbf{S}_p\mathbf{S}_p^t$, so that

$$\begin{aligned}
\mathbf{S}^{pp}\left(\mathbf{S}_p\wedge\mathbf{M}\right) &= \mathbf{S}_p\wedge\mathbf{M} - \mathbf{S}_p\left[\mathbf{S}_p\cdot\left(\mathbf{S}_p\wedge\mathbf{M}\right)\right]\\
&= \mathbf{S}_p\wedge\mathbf{M},
\end{aligned}$$

as, evidently, $\mathbf{A}\cdot(\mathbf{A}\wedge\mathbf{B}) = 0$. □

3. Mechanical networks

So far, the description of the network has been purely geometrical. To give this network a mechanical behavior, we associate with each node P_i a positive number m_i (its *mass*), with each edge p a positive number K_p (the *tensional rigidity*), and with each elbow $[i,(p,q)]$ a positive number $J_{i;(p,q)}$ (the *flexural rigidity*).

Let us then consider a node P_i as a particle of mass m_i subjected to the action of concurrent forces F_i^k. Some of these forces are due to causes external to the network, such as a gravity field or a fluid pressure. The resultant of all these external forces acting on the node P_i is denoted by $\mathbf{F}_i^e(t)$. The other forces, transmitted to P_i by the edges concurring on this node, may be divided into three sets.

Firstly, we find the tensions acting along the edges and resulting from the relative displacements of the two vertices of a same edge. The *tension* in the edge p will be denoted by T_p (this number is positive for a traction, negative for a compression) and the corresponding force is given by $T_p\,\mathbf{S}_p$.

Secondly, there are the bending moments acting on the elbows $[i;(p,q)]$, resulting from a closure or an opening of the angle made by the two edges p and q having i as a common vertex. For $(p,q)\in \mathrm{Ie}^S(i)$, we denote by $\mathbf{M}_{i;(p,q)}$ the vector associated with this bending moment. We refer to these moments as the local bending moments relative to node P_i.

Lastly, we must take into account the actions of the bending moments relative to the nodes P_j linked to the node P_i by an edge p. Such moments, denoted by $\mathbf{M}_{j;(p,q)}$, with $p\in \mathrm{Ie}(i)$ or $q\in \mathrm{Ie}(i)$, are called the neighboring bending moments of node P_i.

Only masspoints and massless rectilinear links have been introduced to describe this network, without any local inertia tensor. As a consequence, the motion of the nodes will obey the dynamical equations obtained by considering the balance of forces applied to each node P_i:

$$\dot{\mathbf{X}}_i = \mathbf{V}_i \qquad m_i\,\dot{\mathbf{V}}_i = \sum_k \mathbf{F}_i^k + \mathbf{F}_i^e. \tag{14}$$

A detailed description of the internal forces F_i^k will be presented below.

3.1. Tensions

The indexes of the edges directed toward or outwards the node P_i are listed in the set $\mathrm{Ie}(i)$.

A tension force applied on a node P_i by an edge $p\in \mathrm{Ie}(i)$ may be written as (Figure 2):

$$\mathbf{F}_i^p = \begin{cases} +T_p\,\mathbf{S}_p & \text{if } p\in \mathrm{Ie}^-(i) \\ -T_p\,\mathbf{S}_p & \text{if } p\in \mathrm{Ie}^+(i) \end{cases} \Longrightarrow \mathbf{F}_i^p = -s_i^p\,T_p\,\mathbf{S}_p \quad \text{for all } p\in \mathrm{Ie}(i).$$

Thus, the node P_i will be subjected to the following tension forces:

$$\sum_{\text{tensions}} \mathbf{F}_i^k = -\sum_{p\in \mathrm{Ie}(i)} s_i^p\,T_p\,\mathbf{S}_p. \tag{15}$$

3.2. Local bending moments

The set $\mathrm{Ie}^S(i)$ contains the couple of edges (p,q) sharing P_i as a common node. The angular displacement of the elbow $[i;(p,q)]$ results in a vectorial bending moment $\mathbf{M}_{i;(p,q)}$, normal to the plane $(\mathbf{S}_p,\mathbf{S}_q)$. This bending moment will subject the nodes P_j and P_k to the forces F_j^i and F_k^i, with $j=a_i^p$ and $k=a_i^q$.

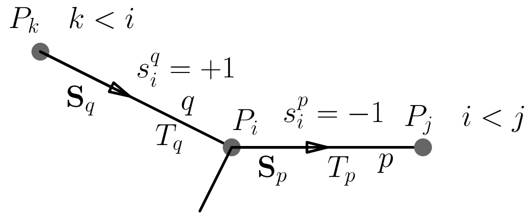

Figure 2. Primary directors and tensions.

The usual convention will be employed to define the orientation of the vectorial bending moment: when the effect of this moment is to *open* the angle (p, q), the set of vectors $\left[\{i \to j\}, \{i \to k\}, \mathbf{M}_{i;(p,q)}\right]$ is right-handed. This moment may be then defined by

$$\mathbf{M}_{i;(p,q)} = M_{i;(p,q)} \frac{\mathbf{S}_{i\to j} \wedge \mathbf{S}_{i\to k}}{\|\mathbf{S}_{i\to j} \wedge \mathbf{S}_{i\to k}\|}. \tag{16}$$

The forces created by $\mathbf{M}_{i;(p,q)}$ on the nodes P_j and P_k can be given as

$$\mathbf{F}_j^i = +\frac{1}{\ell_p}\mathbf{S}_{i\to j} \wedge \mathbf{M}_{i;(p,q)}, \qquad \mathbf{F}_k^i = -\frac{1}{\ell_q}\mathbf{S}_{i\to k} \wedge \mathbf{M}_{i;(p,q)} \tag{17}$$

with $\mathbf{S}_{i\to j} = -s_i^p\,\mathbf{S}_p$ and $\mathbf{S}_{i\to k} = -s_i^q\,\mathbf{S}_q$. Thus,

$$\mathbf{M}_{i;(p,q)} = s_i^p\, s_i^q\, \frac{\mathbf{S}_p \wedge \mathbf{S}_q}{\|\mathbf{S}_p \wedge \mathbf{S}_q\|} M_{i;(p,q)} = \mathbf{B}_{pq}\, M_{i;(p,q)} \tag{18}$$

and

$$\mathbf{F}_j^i = -s_i^p\, \frac{1}{\ell_p}\mathbf{S}_p \wedge \mathbf{M}_{i;(p,q)}, \qquad \mathbf{F}_k^i = +s_i^q\, \frac{1}{\ell_q}\mathbf{S}_q \wedge \mathbf{M}_{i;(p,q)}. \tag{19}$$

As a consequence,

$$\mathbf{F}_j^i = -\frac{s_i^p}{\ell_p}\mathbf{S}_p \wedge \mathbf{B}_{pq}\, M_{i;(p,q)}, \qquad \mathbf{F}_k^i = +\frac{s_i^q}{\ell_q}\mathbf{S}_q \wedge \mathbf{B}_{pq}\, M_{i;(p,q)}.$$

These quantities are sketched in Figure 3. By Newton's third law, the node P_i will be subjected to forces equal and opposite to the forces applied on P_j and P_k by the bending moment acting from P_i. The resultant of these two forces, applied on P_i, is then

$$\mathbf{F}_i^{j,k} = -\left[\mathbf{F}_j^i + \mathbf{F}_k^i\right] = \left(\frac{s_i^p}{\ell_p}\mathbf{S}_p - \frac{s_i^q}{\ell_q}\mathbf{S}_q\right) \wedge \mathbf{B}_{pq}\, M_{i;(p,q)}. \tag{20}$$

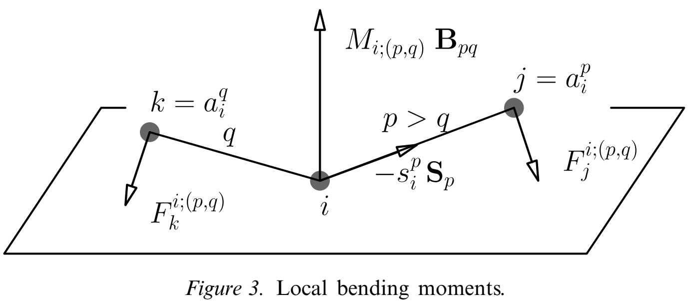

Figure 3. Local bending moments.

Let us introduce the third family of directors $\mathbf{N}_{pq}$ by the relation

$$\mathbf{N}_{pq} = s_i^p \, \mathbf{S}_p \wedge \mathbf{B}_{pq} \tag{21}$$

and let us remark that (since $\mathbf{B}_{qp} = -\mathbf{B}_{pq}$)

$$\mathbf{N}_{qp} = s_i^q \, \mathbf{S}_q \wedge \mathbf{B}_{qp} = -s_i^q \, \mathbf{S}_q \wedge \mathbf{B}_{pq}.$$

Then, we may rearrange (20) to get

$$\mathbf{F}_i^{j,k} = \left(\frac{1}{\ell_p} \mathbf{N}_{pq} + \frac{1}{\ell_q} \mathbf{N}_{qp} \right) M_{i;(p,q)}. \tag{22}$$

The resultant of the forces applied on the node P_i by the local bending moments associated to the elbows $[i;(p,q)]$ may be expressed as

$$\sum_{\text{loc}} \mathbf{F}_i^{\text{loc}} = \sum_{(p,q)\in \text{Ie}^S(i)} \left(\frac{1}{\ell_p} \mathbf{N}_{pq} + \frac{1}{\ell_q} \mathbf{N}_{qp} \right) M_{i;(p,q)}. \tag{23}$$

These directors $\mathbf{N}_{pq}$ and $\mathbf{N}_{qp}$ are the discrete equivalents of the normal vector $\mathbf{n}$ for the curve $(i, s_i^q \, \mathbf{S}_q, -s_i^p \, \mathbf{S}_p)$.

3.3. Neighboring bending moments

The node P_i is connected to its neighbors by the edges $p \in \text{Ie}(i)$. Let $j = a_i^p$ be the second vertex of the edge p. The edge p is one of the two members of a subset of the elbows defined on P_j. The elements of this subset are, for $q \in \text{Ie}(j)$, the elbows $[j;(p,q)]$ $(p>q)$ and $[j;(q,p)]$ $(q>p)$. There are $\text{Card}(\text{Ie}(j)) - 1$ of such elbows.

Let us first consider a couple (p,q) $(p>q)$. The force applied to the node P_i by the bending moment $\mathbf{M}_{j;(p,q)}$ may be written as (see left part of Figure 4)

$$\begin{aligned} \mathbf{F}_i^{j;(p,q)} &= \frac{1}{\ell_p} \mathbf{S}_{j\to i} \wedge \mathbf{M}_{j;(p,q)} = \frac{1}{\ell_p} \left(-s_j^p \, \mathbf{S}_p \right) \wedge \mathbf{M}_{j;(p,q)} \\ &= -\frac{1}{\ell_p} s_j^p \, \mathbf{S}_p \wedge \mathbf{B}_{pq} \, M_{j;(p,q)} = \frac{s_i^p}{\ell_p} \mathbf{S}_p \wedge \mathbf{B}_{pq} \, M_{j;(p,q)}. \end{aligned}$$

In the case (q,p) $(q>p)$, this same force is expressed by (see right part of Figure 4)

$$\begin{aligned} \mathbf{F}_i^{j;(q,p)} &= \frac{1}{\ell_p} \mathbf{M}_{j;(q,p)} \wedge \mathbf{S}_{j\to i} = \frac{1}{\ell_p} \mathbf{M}_{j;(q,p)} \wedge \left(-s_j^p \, \mathbf{S}_p \right) \\ &= \frac{1}{\ell_p} s_j^p \, \mathbf{S}_p \wedge \mathbf{B}_{qp} \, M_{j;(q,p)} = -\frac{s_i^p}{\ell_p} \mathbf{S}_p \wedge \mathbf{B}_{qp} \, M_{j;(q,p)}. \end{aligned}$$

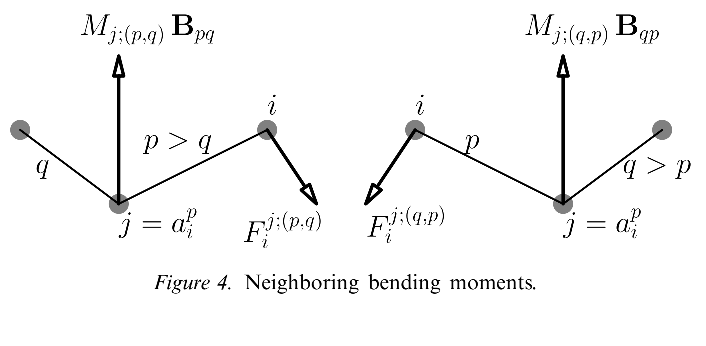

Figure 4. Neighboring bending moments.

In the case (q, p) $(q < p)$, the bending moment is expressed by

$$\mathbf{M}_{j;(q,p)} = \left(s_j^q\,\mathbf{S}_q \wedge s_j^p\,\mathbf{S}_p\right) M_{j;(q,p)} = s_j^q\,s_j^p\,\mathbf{k}_{qp}\,M_{j;(q,p)} \tag{24}$$

and, accordingly,

$$\mathbf{F}_i^{j;(q,p)} = -\frac{1}{\ell_p}\mathbf{S}_{j\to i} \wedge \mathbf{M}_{j;(q,p)} = -\frac{s_j^p}{\ell_p}\mathbf{S}_p \wedge s_j^q\,s_j^p\,\mathbf{k}_{qp}\,M_{j;(q,p)} = -\frac{s_j^q}{\ell_p}\mathbf{S}_p \wedge \mathbf{k}_{qp}\,M_{j;(q,p)} \tag{25}$$

The resultant of all forces applied on the node P_i by the bending moments acting on the neighboring nodes of P_i may then be written down as

$$\begin{aligned}\sum_{\text{neighb}} \mathbf{F}_i^{\text{neighb}} &= \sum_{p\in \mathrm{Ie}(i)} \left[\sum_{(p,q)\in \mathrm{Ie}^S(a_i^p)} \frac{s_i^p}{\ell_p}\mathbf{S}_p \wedge \mathbf{B}_{pq}\,M_{a_i^p;(p,q)} - \sum_{(q,p)\in \mathrm{Ie}^S(a_i^p)} \frac{s_i^p}{\ell_p}\mathbf{S}_p \wedge \mathbf{B}_{qp}\,M_{a_i^p;(q,p)}\right] \\ &= \sum_{p\in \mathrm{Ie}(i)} \frac{s_i^p}{\ell_p}\mathbf{S}_p \wedge \left[\sum_{(p,q)\in \mathrm{Ie}^S(a_i^p)} \mathbf{B}_{pq}\,M_{a_i^p;(p,q)} - \sum_{(q,p)\in \mathrm{Ie}^S(a_i^p)} \mathbf{B}_{qp}\,M_{a_i^p;(q,p)}\right].\end{aligned}$$

4. Dynamical equations

4.1. Positions and velocities

The position $\mathbf{X}_i$ and the velocity $\mathbf{V}_i$ of the node P_i are related by the obvious differential equation

$$\dot{\mathbf{X}}_i = \mathbf{V}_i. \tag{26}$$

Collecting all the forces acting on a node P_i, then Euler's first law of linear momentum results in

$$\begin{aligned} m_i\,\dot{\mathbf{V}}_i = &-\sum_{p\in \mathrm{Ie}(i)} s_i^p\,T_p\,\mathbf{S}_p + \sum_{(p,q)\in \mathrm{Ie}^S(i)} \left(\frac{1}{\ell_p}\mathbf{N}_{pq} + \frac{1}{\ell_q}\mathbf{N}_{qp}\right) M_{i;(p,q)} \\ &+ \sum_{p\in \mathrm{Ie}(i)} \frac{s_i^p}{\ell_p}\mathbf{S}_p \wedge \left[\sum_{(p,q)\in \mathrm{Ie}^S(a_i^p)} \mathbf{B}_{pq}\,M_{a_i^p;(p,q)} - \sum_{(q,p)\in \mathrm{Ie}^S(a_i^p)} \mathbf{B}_{qp}\,M_{a_i^p;(q,p)}\right] + \mathbf{F}_i^e. \end{aligned} \tag{27}$$

A node may be subjected to obey a rheonomic constraint. In this situation, the differential equations concerning that node have to be replaced by

$$\mathbf{X}_i(t) = \mathbf{X}_i^0(t), \qquad \mathbf{V}_i(t) = \dot{\mathbf{X}}_i^0(t), \qquad (X_i^0(t) \text{ given}). \tag{28}$$

4.2. Tensions

Let us consider an edge p. In its unstretched state, the value of the tension in this edge is $T_p = 0$ and the edge length is ℓ_p^0. In a stretched state, a force $T_p\,\mathbf{S}_p$ is acting along the edge, and the length of the edge takes the value ℓ_p. Considering the edge as a spring of stiffness K_p, one may write

$$\frac{\ell_p - \ell_p^0}{\ell_p^0} = \frac{1}{K_p\,\ell_p^0}\,T_p. \tag{29}$$

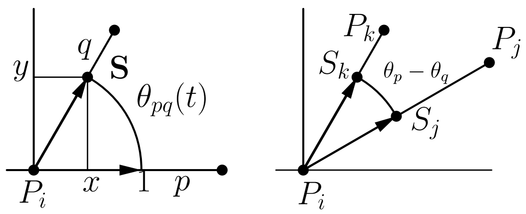

Figure 5. Kinematic of the angle between two edges.

Deriving with regard to time the two sides of this expression, we arrive at $\dot{T}_p = K_p\,\dot{\ell}_p$. Using (10), we obtain the differential equations governing the tensions:

$$\dot{T}_p = K_p\,\mathbf{V}^p\cdot\mathbf{S}_p. \tag{30}$$

4.3. Bending moments

Let $[i;(p,q)]$ be an elbow. In its unstretched state, the angle made by the two edges takes the value θ^0_{pq} and the bending moment acting on this elbow is equal to zero. The displacements of the nodes and edges of the network will modify the angle made by p and q to a value $\theta_{pq}(t)$, and a restoring bending moment $M_{i;(p,q)}$ will then appear, given by

$$M_{i;(p,q)} = -J_{i;(p,q)}\left(\theta_{pq}(t) - \theta^0_{pq}\right), \tag{31}$$

($M_{i;(p,q)} > 0$ when $\theta_{pq} < \theta^0_{pq}$). Let us derive the two sides of this relation with respect to time:

$$\dot{M}_{i;(p,q)} = -J_{i;(p,q)}\,\dot{\theta}_{pq}.$$

We now evaluate the quantity $\dot{\theta}_{pq}$ according to the local kinematics of the two edges p and q.

Let us first consider a local frame with P_i as origin, $-s^p_i\,\mathbf{S}_p$ as first basis vector $\vec{\imath}$ and the plane $(\vec{\imath},\vec{\jmath})$ containing the vector $\mathbf{S}_q$. Let us choose the orientation of $\vec{\jmath}$ so that the triad $[-s^p_i\,\mathbf{S}_p, -s^q_i\,\mathbf{S}_q, \vec{k} = \vec{\imath}\wedge\vec{\jmath}]$ be right-handed:

$$\vec{k} = \frac{(-s^p_i\,\mathbf{S}_p)\wedge(-s^q_i\,\mathbf{S}_q)}{\|(-s^p_i\,\mathbf{S}_p)\wedge(-s^q_i\,\mathbf{S}_q)\|} = s^p_i\,s^q_i\,\frac{\mathbf{S}_p\wedge\mathbf{S}_q}{\|\mathbf{S}_p\wedge\mathbf{S}_q\|} = \mathbf{B}_{pq}. \tag{32}$$

The value of θ_{pq} is directly related to the position of the point S defined by the intersection of the edge q with the unit circle centered on P_i (refer to Figure 5). Let $(x(t), y(t))$ be the coordinates of S. We may write

$$x = \cos\theta_{pq} \quad y = \sin\theta_{pq} \quad \dot{x} = -\sin\theta_{pq}\,\dot{\theta}_{pq} \quad \dot{y} = \cos\theta_{pq}\,\dot{\theta}_{pq},$$

so that

$$\dot{y}\,x - y\,\dot{x} = \cos^2\theta_{pq}\,\dot{\theta}_{pq} + \sin^2\theta_{pq}\,\dot{\theta}_{pq} = \dot{\theta}_{pq}.$$

This last expression may be written vectorially as:

$$\mathbf{S} = \left|\begin{matrix} x \\ y \\ 0 \end{matrix}\right. \quad \dot{\mathbf{S}} = \left|\begin{matrix} \dot{x} \\ \dot{y} \\ 0 \end{matrix}\right. \quad \dot{\theta}_{pq} = \left(\mathbf{S}\wedge\dot{\mathbf{S}}\right)_z = \left(\mathbf{S}\wedge\dot{\mathbf{S}}\right)\cdot\vec{k}. \tag{33}$$

Let us remark now that $\mathbf{S} = -s_i^q\,\mathbf{S}_q$, which results in

$$\dot{\theta}_{pq} = \left(-s_i^q\,\mathbf{S}_q \wedge -s_i^q\,\dot{\mathbf{S}_q}\right) \cdot \mathbf{B}_{pq}, \tag{34}$$

so that, from (9),

$$\dot{\theta}_{pq} = \left(\mathbf{S}_q \wedge \frac{1}{\ell_q}\mathbf{S}^{qq}\mathbf{V}^q\right) \cdot \mathbf{B}_{pq}. \tag{35}$$

It is now possible to generalize the preceeding result to any elbow $[i;(p,q)]$ of the network $\mathcal{G}$. In the plane $(P_i; -s_i^p\,\mathbf{S}_p; -s_i^q\,\mathbf{S}_q)$, we can define an orthogonal right-handed frame with P_i as origin, such that the basis vector $\vec{k}$ and the vector $\mathbf{B}_{pq}$ are oriented in the same way. In the $(\vec{\imath},\vec{\jmath})$ plane, the angle at which the two edges p and q meet at the node P_i has for value $\theta_{pq} = \theta_q - \theta_p$. Then, from the preceeding relation,

$$\dot{\theta}_{pq} = \dot{\theta}_q - \dot{\theta}_p = \left[\left(\mathbf{S}_q \wedge \frac{1}{\ell_q}\mathbf{S}^{qq}\mathbf{V}^q\right) - \left(\mathbf{S}_p \wedge \frac{1}{\ell_p}\mathbf{S}^{pp}\mathbf{V}^p\right)\right] \cdot \mathbf{B}_{pq}. \tag{36}$$

Hence, in this local frame:

$$\dot{M}_{i;(p,q)} = J_{i;(p,q)}\left[\frac{1}{\ell_q}\left(\mathbf{S}^{qq}\,\mathbf{V}^q \wedge \mathbf{S}_q\right) - \frac{1}{\ell_p}\left(\mathbf{S}^{pp}\,\mathbf{V}^p \wedge \mathbf{S}_p\right)\right] \cdot \mathbf{B}_{pq}. \tag{37}$$

Let us permute the triple product:

$$\mathbf{S}^{qq}\mathbf{V}^q \wedge \mathbf{S}_q \cdot \mathbf{B}_{pq} = \mathbf{S}_q \wedge \mathbf{B}_{pq} \cdot \mathbf{S}^{qq}\mathbf{V}^q.$$

The vector $\mathbf{S}_q \wedge \mathbf{B}_{pq}$ is a unit vector, so that we may use the relation (12):

$$\mathbf{S}^{qq}\mathbf{V}^q \cdot \left(\mathbf{S}_q \wedge \mathbf{B}_{pq}\right) = \mathbf{S}^{qq}\left(\mathbf{S}_q \wedge \mathbf{B}_{pq}\right) \cdot \mathbf{V}^q \tag{38}$$

and finally, by using (13), transform this last expression into

$$\mathbf{S}^{qq}\mathbf{V}^q \wedge \mathbf{S}_q \cdot \mathbf{B}_{pq} = \mathbf{S}_q \wedge \mathbf{B}_{pq} \cdot \mathbf{V}^q. \tag{39}$$

In the same way, we may write

$$\mathbf{S}^{pp}\mathbf{V}^p \wedge \mathbf{S}_p \cdot \mathbf{B}_{pq} = \mathbf{S}_p \wedge \mathbf{B}_{pq} \cdot \mathbf{V}^p. \tag{40}$$

Making use of the secondary directors introduced by (21), the differential equation governing the time evolution of the bending moments finally takes the form

$$\dot{M}_{i;(p,q)} = -\,J_{i;(p,q)}\left[\frac{s_i^p}{\ell_p}\mathbf{N}_{pq} \cdot \mathbf{V}^p + \frac{s_i^q}{\ell_q}\mathbf{N}_{qp} \cdot \mathbf{V}^q\right]. \tag{41}$$

4.4. Differential system

Collecting the preceeding results, we may write the first-order differential system governing the motion of a mechanical network as:

$$\dot{\mathbf{X}}_i = \mathbf{V}_i, \tag{42}$$

$$m_i \dot{\mathbf{V}}_i = -\sum_{p\in \mathrm{Ie}(i)} s_i^p T_p \mathbf{S}_p + \sum_{(p,q)\in \mathrm{Ie}^S(i)} \left(\frac{1}{\ell_p}\mathbf{N}_{pq} + \frac{1}{\ell_q}\mathbf{N}_{qp}\right) M_{i;(p,q)} \tag{43}$$

$$+ \sum_{p\in \mathrm{Ie}(i)} \frac{s_i^p}{\ell_p}\mathbf{S}_p \wedge \left[\sum_{(p,q)\in \mathrm{Ie}^S(a_i^p)} \mathbf{B}_{pq} M_{a_i^p;(p,q)} - \sum_{(q,p)\in \mathrm{Ie}^S(a_i^p)} \mathbf{B}_{qp} M_{a_i^p;(q,p)}\right] + \mathbf{F}_i^e$$

$$\dot{T}_p = K_p \mathbf{V}^p \cdot \mathbf{S}_p, \tag{44}$$

$$\dot{M}_{i;(p,q)} = -J_{i;(p,q)} \left[\frac{s_i^p}{\ell_p}\mathbf{N}_{pq}\cdot \mathbf{V}^p + \frac{s_i^q}{\ell_q}\mathbf{N}_{qp}\cdot \mathbf{V}^q\right]. \tag{45}$$

The lengths ℓ_p, the primary directors $\mathbf{S}_p$ and the secondary directors ($\mathbf{B}_{pq}$; $\mathbf{N}_{pq}$) are obtained from the positions $\mathbf{X}_i$ by the relations

$$\ell_p = \|\mathbf{X}_{E(p)} - \mathbf{X}_{B(p)}\|, \tag{46}$$

$$\mathbf{S}_p = \frac{\mathbf{X}_{E(p)} - \mathbf{X}_{B(p)}}{\ell_p}, \tag{47}$$

$$\mathbf{B}_{pq} = s_i^p s_i^q \frac{\mathbf{S}_p \wedge \mathbf{S}_q}{\|\mathbf{S}_p \wedge \mathbf{S}_q\|}, \tag{48}$$

$$\mathbf{N}_{pq} = s_i^p \mathbf{S}_p \wedge \mathbf{B}_{pq}. \tag{49}$$

On each node P_i, the triads $(-s_i^p \mathbf{S}_p, \mathbf{N}_{pq}, \mathbf{B}_{pq})$ and $(-s_i^q \mathbf{S}_q, \mathbf{N}_{qp}, \mathbf{B}_{pq})$ form a set of orthonormal directors. The initial state of the network must be given by the values $(\mathbf{X}_i^0, \mathbf{V}_i^0, T_p^0, M_{i;(p,q)}^0)$ for a time $t = t_0$.

As a particular case, one may consider the standard model for a truss in which the beams can only sustain axial forces. Then, all the moments vanish and the differential system describing the motion of this truss can be written in a much simpler form:

$$m_i \ddot{\mathbf{X}}_i = -\sum_{p\in \mathrm{Ie}(i)} s_i^p T_p \mathbf{S}_p + \mathbf{F}_i^e \qquad \dot{T}_p = K_p \dot{\mathbf{X}}^p \cdot \mathbf{S}_p. \tag{50}$$

The equilibrium state for this truss will be found by solving the system of equations given by

$$-\sum_{p\in \mathrm{Ie}(i)} s_i^p K_p (\ell_p - \ell_p^0)\mathbf{S}_p + \mathbf{F}_i^e = 0. \tag{51}$$

Such equations are well known and could have been derived directly in a simpler way through the Lagrange function associated with the network.

5. A numerical example

An example of a mechanical network which evolves according to the differential laws (42–45) will now be presented. This example is taken from Shi *et al.* [5] who describe a benchmark for testing dynamic analysis packages that deals with flexible bodies. A beam is built into a flexible shaft that is driven with an angular displacement $\theta(t)$ around the z-axis. Assuming

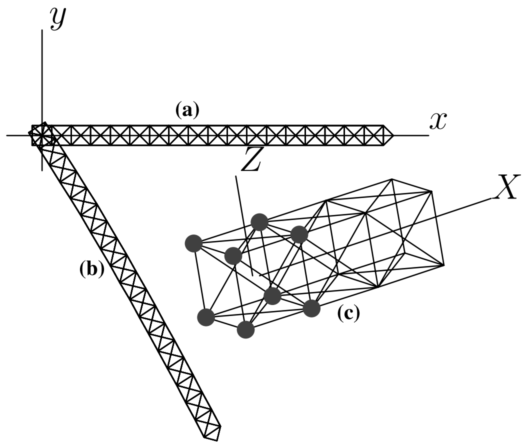

Figure 6. Initial (a) and moving (b) network. Driving nodes on (c).

zero gravity, one has to compute the beam's tip deformation in its body frame (X, Y). This beam is modeled by a network of 77 nodes and 296 edges represented in Figure 6. The first eight nodes are constrained to rotate around the z-axis with the same angular displacement $\theta(t)$ given by

$$\theta(t) = \begin{cases} \frac{\omega_s}{T_s}\left[\frac{t^2}{2} + \left(\frac{T_s}{2\pi}\right)^2\left(\cos\left(\frac{2\pi t}{T_s}\right) - 1\right)\right] & t < T_s \\ \omega_s\left(\frac{T_s}{2}\right) & t \geq T_s \end{cases} \tag{52}$$

with $\omega_s = 6{\cdot}0\,\mathrm{rad/s}$ and $T_s = 15\,\mathrm{s}$. These eight nodes behave collectively as a rigid box clamped to a rotating z-axis driving the flexible part of the network. This network has a total length $L = 10\,\mathrm{m}$ and a square cross-section $b = 0{\cdot}1\,\mathrm{m}$ (in Figure 6, the cross-sections have been oversized by a factor of five).

A numerical value must be assigned to each of the physical constants m_i, K_p and $J_{i;(p,q)}$. By now, no general method is known which will ensure a network motion reproducing the deformation of a corresponding three-dimensional continuum media. In that example, preliminary results pertaining to one-dimensional networks will be used. We consider each edge p to be a local beam for which the area of the cross-section is A_p and the area's moment of inertia is I_p. This local beam is assumed to be made of a material with an elasticity modulus E_p and a volumic mass ρ_p. Then, the mass m_i, the edge stifness K_p and the elbow stiffness $J_{i;(p,q)}$ are computed by the following relations

$$m_i = \frac{1}{2}\sum_{p \in \mathrm{Ie}(i)} \rho_p\, A_p\, \ell_p^0, \quad K_p = \frac{E_p\, A_p}{\ell_p^0}, \quad J_{i;(p,q)} = \frac{E_p\, I_p + E_q\, I_q}{\ell_p^0 + \ell_q^0}.$$

The material properties will be taken from [5]: $E = 7{\cdot}0 \times 10^{10}\,\mathrm{Pa}$, $I = 2{\cdot}0 \times 10^{-7}\,\mathrm{m}^2$, $A = 4{\cdot}0 \times 10^{-4}\,\mathrm{m}^2$, total mass $M = 12\,\mathrm{kg}$. The beam has been discretized by eighteen boxes made of twelve lateral edges linked by four transversal edges. For all edges, square cross-sections of area $A/12$ have been assumed. Then, the physical constants in this network are

$$0{\cdot}106 \leq m_i \leq 0{\cdot}160 \quad 4{\cdot}1 \times 10^6 \leq K_p \leq 2{\cdot}5 \times 10^7 \quad 2{\cdot}3 \times 10^3 \leq J_{i;(p,q)} \leq 8{\cdot}5 \times 10^3$$

In this example, the differential system (42–45) is large (77 nodes, 296 edges and 2010 elbows resulting in 2383 differential relations) and stiff. Explicit numerical methods

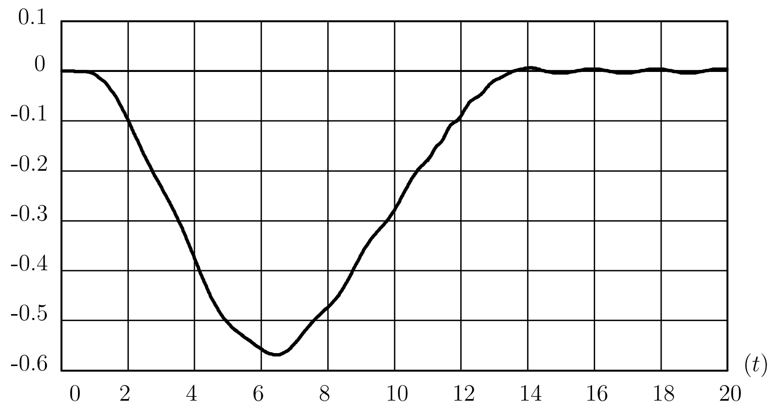

Figure 7. Tip deflection of the motion-driven beam.

would have to use a very small time step to ensure stability of the scheme. In order to overcome this problem, we turned to the use of semi-implicit numerical derivatives which will be described precisely elsewhere. Briefly said, these derivatives are obtained in three steps. First, an explicit step gives the differentials $\delta\mathbf{X}_i$ of the positions. Then, the new positions are used to set up a linear system from which are obtained implicitly the differentials of the tensions δK_p and the differentials of the bending moments $\delta J_{i;(p,q)}$. The third step computes the velocity differentials $\delta\mathbf{V}_i$ from the relation (43). Semi-implicit numerical schemes of arbitrary order may be obtained simply by using semi-implicit derivatives in place of ordinary derivatives in the usual Runge-Kutta schemes. We have used for the computations a semi-implicit Runge-Kutta scheme of order 4, and a time step $\tau = 1 \times 10^{-3}$ (s).

The time evolution of the tip deflection $Y(t) = -\sin\theta(t)\,x_{nn}(t) + \cos\theta(t)\,y_{nn}(t)$ is plotted on the Figure 7 (the node nn is at the tip of the beam). The largest deflection $Y_{\max} = -0{\cdot}568$ is obtained for $t = 6{\cdot}46$, in close agreement with $t \approx 6{\cdot}5$, $Y_{\max} = -0{\cdot}573$ as given by [5]. One may notice, however, a slightly larger delay for the first deflections to appear and a lower time of recovery in the phase of decreasing angular acceleration. This lack of stiffness could be overcome by adjusting the flexural stiffness coefficients J_i or by adding internal edges in order to stiffen the elementary boxes.

6. Conclusion

The differential equations governing the motion of three-dimensional mechanical networks have been obtained. The mechanical components of the state vector include not only the tensions that belong to the edges but also the bending moments defined at the junction of two edges. With this added feature, these networks may then be used to model the elastic behavior of a deformable continuum. It is shown that the time evolution of the state vector is governed by a set of triads of mutually orthogonal directors depending on the spatial orientation of the edges of these networks. A planar rotating beam has been used as a numerical example. The stiffness coefficients have been computed on the assumption that each edge should react as an elementary beam and the results show good agreement with those reported in [5]. However, at the time of this writing, no proof of convergence is available in support of this assumption.

This work may be related also to the field of multibody dynamics [6] from which the numerical example was taken. The equations of motion for networks of rigid bodies were considered in the work of Wittenburg [7]. The differential system (42–45) may be used to model the dynamics of a system of rigid, elastic and flexible bodies connected together by various types of joints. A rigid body has to be approximated by a system of rigidly connected mass-points, with consideration given to the total mass and to the inertia tensor of this corresponding subnetwork. Elastic bodies (such as beams) are approximated by "elastic" subnetworks (*i.e.*, involving tensions and bending moments). Flexible structures (strings or draperies) are approximated by subparts for which the flexural rigidities are taken to be zero. In the future, it is desirable to apply the framework described in this work to these various situations.

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